Variational principle of Hall MHD

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Introduction

The Hall effect, which is written by higher order derivative term, is the singular perturbation to the MHD equations.

A variational principle that solves minimum energy state under two helicities constraints is known to be an ill-posed problem because of the singular perturbation of the Hall term.

We studied how the ill-posedness appears in the analysis of equilibrium (minimum energy state) and Lyapunov stability in Hall MHD system.
Hall term and singular perturbation

Hall-MHD equations

\[
\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = (\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p + \frac{1}{Re} \Delta \mathbf{V}, \quad (1)
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times [(\mathbf{V} - \epsilon \nabla \times \mathbf{B}) \times \mathbf{B}] + \frac{1}{Rm} \Delta \mathbf{B}, \quad (2)
\]

\[
\nabla \cdot \mathbf{V} = 0. \quad (3)
\]

Scaling coefficient \( \epsilon = \frac{l_i}{L_0} \) is a measure of the ion skin depth \( l_i = c/\omega_{pi} = V_A/\omega_{ci} = \sqrt{M/\mu_0 n e^2} \).

The only addition to the standard MHD is the Hall current term \( \epsilon (\nabla \times \mathbf{B}) \times \mathbf{B} \) in (2).

Mathematically \( \epsilon \) is a singular perturbation parameter, since it multiplies the highest derivative term (in the ideal limit \( 1/Re = 1/Rm = 0 \)).
Minimum energy \( (E) \) with constraints of helicities \((H_1, H_2)\) are characterized by the following ill-posed variational principle.

\[ \delta (E - \mu_1 H_1 - \mu_2 H_2) = 0 \quad (4) \]

\[
E = \frac{1}{2} \int_{\Omega} (|B|^2 + |V|^2) \, dx, \quad (5)
\]

\[
H_1 = \frac{1}{2} \int_{\Omega} A \cdot B \, dx, \quad (6)
\]

\[
H_2 = \frac{1}{2} \int_{\Omega} (A + \epsilon V) \cdot (B + \epsilon \nabla \times V) \, dx. \quad (7)
\]
Euler-Lagrange equation

Double Beltrami field

\[ B = C_+ G_+ + C_- G_-, \quad V = (\lambda_+ - \mu_1)C_+ G_+ + (\lambda_- - \mu_1)C_- G_-, \]

where \( \nabla \times G_\pm = \lambda_\pm G_\pm \) and \( \lambda_\pm = \frac{1}{2} \left[ (\mu_2^{-1} + \mu_1) \pm \sqrt{(\mu_2^{-1} - \mu_1)^2 - 4} \right]. \)

\[ \Downarrow \]

Minimum energy state

One of \( \lambda_\pm \) approaches to \( \infty \) and corresponding \( C_\pm \) becomes 0. One of Beltrami fields \( (G_\pm) \) becomes singular and vanish.

\( (\text{Almost every where zero}) \)

\[ \Rightarrow \text{Taylor Relaxed State} \quad \nabla \times B = \lambda_0 B, \quad V = 0. \]

(\( \lambda_0 \) is the minimum eigenvalue of the curl operator.)

In the dissipative system, the kinetic (flow) energy may concentrates in the small scale (singular part) and dissipates intensively.
Example of “ill-posed variational principle”

Minimizer of the energy of a real function \( u(x) \) defined on \((0, \pi)\) such that \( u(0) = u(\pi) = 0; \) \( F_1(u) = \int_0^\pi u^2 \, dx. \)

Without any constraint, the minimizer is the trivial \( u(x) = 0. \)

A “fragile” constraint is imposed; \( F_2(u) = \int_0^\pi \left( \frac{du}{dx} \right)^2 \, dx = 1. \)

The variational principle \( \delta(F_1 - \nu F_2) = 0 \) yields \( -\frac{d^2}{dx^2} u = \frac{1}{\nu} u. \)

The solution must be one of the eigenfunctions \( u(x) = C \sin(\alpha x) \quad (\alpha = \pm \sqrt{1/\nu} = \pm 1, \pm 2, \pm 3 \ldots). \) \( \quad (8) \)

\( F_2 = 1 \) leads to \( C^2 = 2/(\pi \alpha^2) = 2\nu/\pi, \) and \( F_1(u) = 1/\alpha^2 = \nu. \)

The largest eigenvalue \( (\alpha^2 \rightarrow \infty) \) gives the minimum of \( F_1 \)

\( \inf F_1(u) = 0, \) i.e. \( u(x) = 0. \) The fragility of \( F_2(u) \) is due to the fact that it includes a higher-order derivative in comparison with the target functional \( F_1(u). \)
Numerical simulation

We solve the 2-D Hall MHD equations by numerically.

*S. Ohsaki, Phys. Plasmas 12, 032306 (2005).*

Typical time evolution of current governed by Hall MHD

Turbulence is developed and the fields are relaxed into a large scale structure like the MHD evolution.
Time evolution of $E$, $H_1$ and $\hat{H}_2 = (H_2 - H_1)/\varepsilon$ (; cross helicity in MHD limit) in (a) MHD and (b) Hall MHD.

If $H_1$ and/or $\hat{H}_2$ do not change much while $E$ diminishes, the “selective dissipation” of $E$ may yield the minimizer of $E$ under the constraints on the approximate constants of motion.
The kinetic energy dissipates much faster and remains almost no-flow in the final state in the Hall MHD system.
Energy spectrum $E(k_x)$ in the MHD and Hall-MHD models

The Hall term produces small scale structures effectively, since the energy density at high $k_x$ of Hall MHD is larger than that of MHD.
In a non-Hermitian system, energy (Hamiltonian) is not the determinant of dynamics. The stability analysis, then, requires a wider framework.

The notion of “Lyapunov function” is a natural extension of Hamiltonians.

Some constants of motion that give bounds for all fluctuations can be found in special class of double Beltrami field, which is a relaxed state with shear flow in Hall MHD.
Lyapunov stability of MHD flows (Beltrami field)


The variation

\[
\delta(E^M - \mu_1 H_1^M - \mu_2 H_2^M) = 0,
\]

where

\[
E^M = \|B\|^2 + \|V\|^2 \equiv \int B^2 + V^2 \, dx,
\]

\[
H_1^M = (A, B) \equiv \int A \cdot B \, dx,
\]

\[
H_2^M = 2(V, B),
\]

gives single Beltrami field defined by

\[
(1 - \mu_2^2) \nabla \times B = \mu_1 B, \quad V = \mu_2 B.
\]

Writing \(B = B_0 + \tilde{B}\) and \(V = V_0 + \tilde{V}\), the variation (9) implies

\[
G(\tilde{B}, \tilde{V}) = \|\tilde{B}\|^2 + \|\tilde{V}\|^2 - \mu_1 (\tilde{A}, \tilde{B}) - 2\mu_2 (\tilde{V}, \tilde{B})
\]

is a constant of motion of the perturbations.
Using coerciveness condition (Poincaré type inequality)

\[(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}) \leq |\sigma|^{-1}\|\tilde{\mathbf{B}}\|^2,\]  
\[\text{(12)}\]

where \(|\sigma| = \min_j |\sigma_j| \) \([\sigma_j \ (j = 1, 2, \cdots) \) are the eigenvalues of the self-adjoint curl operator], and

\[2(\tilde{\mathbf{V}}, \tilde{\mathbf{B}}) \leq \alpha \|\tilde{\mathbf{V}}\|^2 + \alpha^{-1}\|\tilde{\mathbf{B}}\|^2 \quad (\forall \alpha > 0),\]

we get

\[G(\tilde{\mathbf{B}}, \tilde{\mathbf{V}}) \geq(1 - \alpha|\mu_2|)\|\tilde{\mathbf{V}}\|^2 + \left(1 - \frac{|\mu_2|}{\alpha} - \frac{|\mu_1|}{|\sigma|}\right)\|\tilde{\mathbf{B}}\|^2,\]

\[\text{(13)}\]

where \(G(\tilde{\mathbf{B}}, \tilde{\mathbf{V}})\) is a constant determined by initial condition. \(G\) gives bounds of \(\|\tilde{\mathbf{B}}\|^2\) and \(\|\tilde{\mathbf{V}}\|^2\) when

\[\mu_2^2 < 1, \quad |\lambda| \equiv \frac{|\mu_1|}{1 - \mu_2^2} < |\sigma|.\]

\[\text{(14)}\]

(where \(\nabla \times \mathbf{B} = \lambda \mathbf{B}\))
Lyapunov Stable

\[ |\mu_2| = \frac{|V|}{|B|} \]

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\[ (\nabla \times B = \lambda B) \]

Lyapunov Stable

\[ |\sigma| \]

\[ \lambda \]
Lyapunov stability of Hall MHD flows (double Beltrami field)

The variation (4) characterizing the double Beltrami field implies that the integral

\[ G(\tilde{B}, \tilde{V}) = \|\tilde{B}\|^2 + \|\tilde{V}\|^2 - \mu_1 (\tilde{A}, \tilde{B}) - \mu_2 (\tilde{A} + \tilde{V}, \tilde{B} + \nabla \times \tilde{V}) \]

is a constant of motion of the perturbations.

However, reflecting the fact that the functional \( F = E - \mu_1 H_1 - \mu_2 H_2 \) is not a coercive form in the energy \( (L^2) \) norm, this constant of motion \( G(\tilde{B}, \tilde{V}) \) does not yield a bound for energy of perturbations. \cite{Holm1987}

\[ [D.D. Holm, Phys. Fluids. 30, 1310 (1987).] \]

\[ \downarrow \]

We have to find a constant of motion that contains positive definite higher order derivatives; enstrophy \( (\|\nabla \times V\|^2) \) order constant in the present case.
We can find the Lyapunov functions for two special configurations of the double Beltrami fields;

1. Longitudinal flow system with a spiral magnetic field,
2. Longitudinal magnetic field system with a spiral flow.

*S. Ohsaki, and Z. Yoshida, Phys. Plasmas 10, 3853 (2003).*

### Lyapunov stability condition

\[ \lambda_\pm = -\lambda_\pm \]

\[ \lambda_\pm^2 < \mu^2 - 1 \quad \text{and} \quad \lambda_\pm^2 < \mu^2 \]

\( \mu^2 \) is the smallest eigenvalue of the Laplacian \( -\Delta \).
Summary

We discussed the Hall effect (singular perturbation) on variational principle.

The variational principle that solves the minimum energy state under constraints on $H_2$ is an ill-posed problem. As a result, $H_2$ changes much faster than $E$, implying that $H_2$ is primarily dominated by small-scale components. Because of the loss of $H_2$, the “relaxed state” does not contain a flow.

In the stability analysis, since a functional characterized by the variational principle is unbounded due to the Hall term, we can not get Lyapunov stability condition. However we have found enstrophy order constants that yield a bound any fluctuation (not only exponential but also secular growth) for two special configurations of the double Beltrami fields.